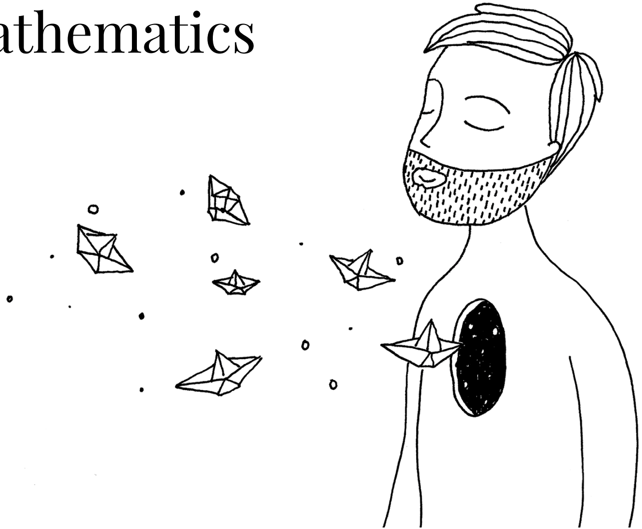
4509 - Bridging Mathematics

Sets



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Sets

A **set** is a collection of **elements**. We say that element a belongs to the set A, if a is contained in A. We denote that: $a \in A$. If a does not belong to A, we write $a \notin A$.

There are two very important sets: The Universal set $\mathcal{U}$ and the empty set $\emptyset$.

Sets

$\mathcal{U}$ is defined as the set that contains all the *relevant* elements.

$\emptyset$ is defined as the set that does not contain any element.

Sets

Definition

Two sets A and B are said to be **equal** if

$$\forall x, \quad x \in A \Leftrightarrow x \in B$$

Definition

B is a **subset** ($\subseteq$) of A if:

$$\forall b \in B \Rightarrow b \in A$$

Quick Quiz - 5 minutes

Show that if A and B are equal, then $A \subseteq B$ and $B \subseteq A$.

Sets

Definition

The **power** of a set is the set of all its subsets.

$$\mathcal{P}(A) := \{B | B \subseteq A\}$$

Note that the elements of A do not belong to $\mathcal{P}(A)$, however, the sets containing a single element do. In other words the **set** $\{a\}$ is different from the **element** a.

Definition

The **cardinality** of a set is the number of elements it contains. It is denoted as $\#A$.

Sets

Let's consider an example. Let $A = \{1, 2, 3\}$.

- $\mathcal{P}(A) = \{\emptyset, A, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$.
- $\# A = 3$, and $\# \mathcal{P}(A) = 8$.

In general, if $\# A$ is finite, then $\# \mathcal{P}(A) = 2^{\# A}$.

Sets

Definition

A set A is to be called finite, if $\#A$ is finite, and infinite otherwise.

Consider the set $\mathbb{N}$, the natural numbers. $\#\mathbb{N} = \infty$. This infinite set is special, because you can count it, from zero to infinity. We call this cardinality *aleph zero*, $\aleph_0$, and the set is **countable**.

Now consider the set $\mathbb{R}$, which you know as the real numbers. Again, $\#\mathbb{R} = \infty$. This cardinal is called *continuum*, and denoted with c. The sets with this cardinality are considered **uncountable**.

Quick Quiz - 5 minutes

Classify the cardinality of the following sets:

Set	Cardinality
$\mathbb{Z}$	$\aleph_0$
$\mathbb{Q}$	$\aleph_0$
$f(x) = a + bx, x \in \mathbb{R}^+$	c

Sets

Definition

Consider A and B to be subsets of $\mathcal{U}$.

1. The union between A and B:

$$A \cup B := \{c| \quad c \in A \quad \vee \quad c \in B\}$$

2. The intersection between A and B:

$$A \cap B := \{c| \quad c \in A \quad \wedge \quad c \in B\}$$

3. The difference between A and B:

$$A \setminus B := \{c| \quad c \in A \quad \wedge \quad c \notin B\}$$

4. The complement of A,

$$A^c := \mathcal{U} \setminus A$$

Sets

If two sets (A and B) are such that there is a valid operation between their elements, say "*ope*", then:

$$A \quad ope \quad B := \{c | c = a \quad ope \quad b, \quad a \in A \quad b \in B\}$$

Example: $A = \{1, 2\}$, $B = \{3, 4, 7\}$,

- $A + B = \{4, 5, 6, 8, 9\}$
- $A - B = \{-6, -5, -3, -2, -1\}$

Sets

Definition

A and B are **disjoint** if

$$A \cap B = \emptyset$$

Definition

A **partition** P of a set A, is a set of k nonempty subsets of A, $\{A_i\}_{i=1}^{k}$, such that:

- A_i and A_j are disjoint for any $i \neq j$.
- $\bigcup_{i=1}^{n} A_i = A$

Sets

Definition

Let $a \in A$, and $b \in B$ (nothing prevents A being equal to B), the **ordered pair** (a, b) is defined as:

$$(a, b) := \{a, \{a, b\}\}$$

Note that while $\{a, b\} = \{b, a\}$ (both sets have the same elements), $(a, b) \neq (b, a)$.

Sets

Definition

The **product** of two sets is defined as:

$$A \times B := \{(a, b) | a \in A \quad b \in B\}$$

Definition

The ordered **n-tuple** of A is:

$$A^n := \underbrace{A \times A \times A \times \ldots \times A}_{\text{n times}}$$

Numbers and Sets

- The naturals, $\mathbb{N} := \{0, 1, 2, 3, \dots\}$
- The integers, $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$
- The rationals, $\mathbb{Q} := \left\{ \frac{p}{q} | p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\} \right\}$
- The reals, $\mathbb{R} := \mathbb{Q} \cup I$, with I being the irrationals.

It holds that

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

Working on the Real Numbers

Let $a < b$, and $a, b \in \mathbb{R}$.

1. $[a, b]$ is called closed interval. This set contains all the real numbers between a and b, with a and b included.
2. $(a, b] := [a, b] \setminus \{a\}$.
3. $[a, b) := [a, b] \setminus \{b\}$.
4. $(a, b) := [a, b] \setminus \{a, b\}$. This is called an open interval. It is also the *interior* of $[a, b]$.

Working on the Real Numbers

If a subset A of $\mathbb{R}$ is such that $A := \{x \in \mathbb{R} | x \leq a\}$ we can write it as $(-\infty, a]$.

The side where the ∞ is, is always "open".

Working on the Real Numbers

Definition

Let $x \in \mathbb{R}$, the module of x is,

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{otherwise} \end{cases}$$

Working on the Real Numbers

Definition

Let $A \subseteq \mathbb{R}$,

1. A is said to be bounded from above if

$$\exists M \in \mathbb{R} \quad s.th. \quad \forall a \in A, a \leq M$$

   M is an upper bound of A.

2. A is said to be bounded from below if

$$\exists M \in \mathbb{R} \quad s.th. \quad \forall a \in A, a \geq M$$

   M is a lower bound of A.

3. A is said to be bounded if it is bounded from above and from below.

Working on the Real Numbers

Let $A \subseteq \mathbb{R}$ be bounded.

Definition

- The **supremum** of A ($sup(A)$) is the lowest of its upper bounds.
- The **infimum** of A ($inf(A)$) is the highest of its lower bounds.

If $sup(A) \in A$, it is called the **maximum** of A ($max(A)$).
If $inf(A) \in A$, it is called the **minimum** of A ($min(A)$).